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CROSSED PRODUCTS OF TYPE I AF ALGEBRAS BY ABELIAN GROUPS

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ABSTRACT

Let (G, A, α) be a separable C^*-dynamical system, with G abelian, and let Γ denote the dual group of G. We characterize the Γ-invariant ideals of the crossed product algebra $G \times_\alpha A$, and use this characterization to prove that if in addition G is compact and A is type I AF, then $G \times_\alpha A$ is AF also. Finally, assuming G is discrete abelian and both A and $G \times_\alpha A$ are type I, we determine necessary and sufficient conditions, in terms of A and the isotropy subgroups for the action of G on $\hat{A}$, for $G \times_\alpha A$ to be AF.

§1. Introduction

It is an open question whether the C^*-crossed product $G \times_\alpha A$ of an AF C^*-algebra A by a finite group G is AF. In fact, the question was first raised by Bratteli, in a preliminary version of [2], for compact group actions on an AF algebra. It was raised again, by Effros in [7, problem 7, page 30], in slightly altered form. Effros asked whether the fixed point algebra of a finite group of automorphisms G on an AF algebra is again an AF algebra, and remarked that this is unknown even if G has 2 elements. In view of the well known relationship between the crossed product algebra and the fixed point algebra, Effros' question may as well have been asked about the crossed product algebra.

In the case of separable C^*-dynamical systems, we prove in this paper that Bratteli's question has an affirmative answer in the special case that A is a type I AF algebra and G is a compact abelian group. The type I hypothesis ensures that $\hat{A}$, the space of unitary equivalence classes of irreducible representations of

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$\hat{A}$, equipped with the hull-kernel topology, has enough of a Hausdorff like structure so that by a composition series argument we may without loss of generality assume it, and hence also $\hat{A}/G$, is Hausdorff. The hypothesis that G is abelian allows us to employ Takai duality to prove that the map of $(G \times_\alpha A)^\wedge$ onto $\hat{A}/G$ which assigns to each irreducible representation of $G \times_\alpha A$ the orbit in $\hat{A}$ on which it is based (see §2 for details as well as for a discussion in more generality) is open. Knowing both that the above map is open and that its range is Hausdorff, we can use Theorem 4 of [14] to represent $G \times_\alpha A$ as a continuous field of C^*-algebras, analyze the fiber algebras and the base space, and prove our main result.

In §2 we consider separable C^*-dynamical systems (G, A, α) with G abelian. We obtain apparently new results, by representation-theoretic methods, on the structure of those ideals of the crossed-product algebra $G \times_\alpha A$ which are invariant under a certain subgroup of the dual group Γ of G, with respect to the dual action of Γ on $G \times_\alpha A$. By letting the subgroup be Γ itself, we obtain as Corollaries that the map of $\mathrm{PR}(G \times_\alpha A)$ into the quasi-orbit space $(\mathrm{PR}(A)/G)^\sim$ (see §2 for a detailed discussion) which assigns to each primitive ideal of $G \times_\alpha A$ the quasi-orbit in $\mathrm{PR}\, A$ on which it is based, is open, and induces a homeomorphism between the quasi-orbit spaces $(\mathrm{PR}(G \times_\alpha A)/\Gamma)^\sim$ and $(\mathrm{PR}(A)/G)^\sim$. We also verify, even when A is non-type I, that every quasi-orbit in $\mathrm{PR}\, A$ does indeed have a primitive ideal of $G \times_\alpha A$ "living over it".

In §3 we consider the case of a compact abelian group G acting on a type I AF algebra A, and prove that the crossed product algebra $G \times_\alpha A$ is AF. We show, by a simple topological argument, that A has a non-zero G-invariant ideal I such that $\hat{I}$, and hence also $\hat{I}/G$, are Hausdorff. We actually need only the weaker result that $\hat{I}/G$ is Hausdorff, and this result actually follows from a deep theorem of Glimm [10, Theorem 1, Condition 5]. However, in our case of compact G, the stronger result is so much easier to prove that we do so directly. The proof of our main theorem then follows from composition series arguments, known results about AF algebras, the results of §2 applied to represent $G \times_\alpha I$ (I as above) as an algebra of continuous sections, and results of Phil Green which describe the structure of the resultant fiber C^*-algebras.

Finally, in §4, we apply the results of §3 and Takai duality to the case of a discrete abelian group G acting on a C^*-algebra A. Under the assumption that both A and $G \times_\alpha A$ are type I, we prove that $G \times_\alpha A$ is AF if and only if A is AF and for each $\pi \in \hat{A}$, $\hat{G}_\pi$, the dual of the isotropy group G_π, is totally disconnected. This last result is perhaps of extra interest due to Proposition 4.3.2 of [4] which states that the crossed product of any unital C^*-algebra by the integers is not AF. It follows that the integers cannot act freely and smoothly on a type I unital AF algebra.

As a reference for all unexplained terminology and results concerning crossed product algebras, including Takai duality and induced representations, see [17] and/or [12]. Throughout this paper, all groups, C^*-algebras, Hilbert spaces, etc., are assumed to be separable. There is a 1–1 correspondence between representations L of the crossed product algebra $G \times_\alpha A$ and covariant pairs of representations $\langle V, \pi \rangle$ of the C^*-dynamical system (G, A, α), and we often blur the distinction by writing $L = \langle V, \pi \rangle$.

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§2. On invariant ideals of $G \times_\alpha A$

Let (G, A, α) denote a separable C^*-dynamical system, with G abelian, and let Γ denote the dual group of G. It was proven in [16, Lemma 6.1] that the existence of non-trivial G-invariant ideals of A is equivalent to the existence of non-trivial Γ-invariant ideals of the crossed product C^*-algebra $G \times_\alpha A$. The proof proceeds by explicitly constructing, for a given non-trivial Γ-invariant ideal of $G \times_\alpha A$, an associated non-trivial G-invariant ideal of A, and by appealing to Takai duality for the converse. More generally, let H be a closed subgroup of G. The action of G on A extends naturally to an action of G on the crossed product algebra $H \times_\alpha A$, and in [15, Proposition 2.5] it is proven that the existence of non-trivial G-invariant ideals of $H \times_\alpha A$ is equivalent to the existence of non-trivial ideals of $G \times_\alpha A$ invariant under the action of $H^\perp$, the annihilator of H in Γ. In [15] the proof proceeds by explicit construction in both directions, and assigns to a non-trivial $H^\perp$-invariant ideal J of $G \times_\alpha A$ a non-trivial G-invariant ideal $I_H(J)$ of $H \times_\alpha A$, and conversely, to a non-trivial G-invariant ideal N of $H \times_\alpha A$ a corresponding non-trivial $H^\perp$-invariant ideal $\operatorname{Ext} N$ of $G \times_\alpha A$. Furthermore, it is proven in [15, Proposition 2.6] that for j and N as above, the relations $\operatorname{Ext}(I_H(J)) \subseteq J$ and $I_H(\operatorname{Ext} N) \supseteq N$ hold, with equality for G discrete [15, Proposition 2.7 and 2.8]. The question of equality or inequality in general is not raised.

In [12, Proposition 9] two pairs of maps are defined relating ideals of $H \times_\alpha A$ with ideals of $G \times_\alpha A$, in the more general context of twisted crossed product algebras and not necessarily abelian groups G. The relationship between several of these maps is also discussed. Now it is clear by comparing [12, Proposition 9] with [15, Lemma 2.2] that the map Ext of [15] is identical to the map Ex of [12] which by [12, Proposition 13] is identical, for G abelian and for G-invariant ideals of $H \times_\alpha A$, to the map Ind. Note that Proposition 13(ii) of [12] only applies to the case $H = N_{\mathcal{T}}$ (see [12] for notation and a complete discussion of the twisted crossed product construction) and to G-invariant ideals I of A. However, as in pages 198–199 of [12], $G \times_\alpha A$ can be considered as a twisted crossed product formed from the action of G on $H \times_\alpha A$, with twisting subgroup H, so that Proposition 13(ii) of [12] does actually apply. The map Ind has a representation-theoretic interpretation. If N is an ideal of $H \times_\alpha A$ and L a representation with kernel N, then Ind N is the kernel of the induced representation Ind L of $G \times_\alpha A$ [12, Proposition 9(iii)]. It is also fairly clear that for an $H^\perp$-invariant ideal J of $G \times_\alpha A$, the ideal $I_H(J)$ defined in Lemma 2.1 of [15] is identical to Res J as defined in Proposition 9(ii) of [12], although we omit details as we do not explicitly need this fact. The map Res also has a representation-theoretic interpretation. If J is an ideal of $G \times_\alpha A$ and $L = \langle V, \pi \rangle$ is a representation of $G \times_\alpha A$ with kernel J, then Res J is the kernel of the representation $L|_H = \langle V|_H, \pi \rangle$ of $H \times_\alpha A$.

It follows from the above discussion and Proposition 11(ii) of [12] that for a G-invariant ideal N of $H \times_\alpha A$, Res Ind $N = N$ so that in fact Proposition 2.7 of [15] holds without the hypothesis that G be discrete. What does not seem to follow formally from the manipulations of [12], however, is the following

Lemma 2.1. *Let (G, A, α) be a separable C^*-dynamical system with G abelian, and let H be a closed subgroup of G. If J is an $H^\perp$-invariant ideal of $G \times_\alpha A$, then* Ind(Res J) $= J$.

Proof. We shall give a completely self-contained proof, using the representation-theoretic interpretations of Res and Ind. Accordingly, let $L = \langle V, \pi \rangle$ be a representation of $G \times_\alpha A$ on a Hilbert space $\mathcal{H}_L$, with kernel $L = J$. For each $\chi \in H^\perp$, $J = \chi J =$ kernel χL, where χL is the representation of $G \times_\alpha A$ corresponding to the covariant pair $\langle \chi V, \pi \rangle$. It follows that J is the kernel of the direct integral representation $\int_{H^\perp} \oplus \langle \chi V, \pi \rangle d\chi$ on $L^2(H^\perp, \mathcal{H}_L)$. As $H^\perp$ is naturally the dual group of G/H, and the direct integral of the characters of G/H (viewed as a representation of G) is unitarily equivalent, via the Fourier transform, to the quasi-regular representation W of G on $L^2(G/H)$, we have $J =$ kernel$\langle W \otimes V, I \otimes \pi \rangle$ on $L^2(G/H, \mathcal{H}_L)$. The proof clearly follows once we verify that $\langle W \otimes V, I \otimes \pi \rangle$ is unitarily equivalent to Ind$(\langle V|_H, \pi \rangle)$. This latter fact is to be expected since $W = \text{Ind}(T)$, where T is the trivial representation of H, and a result of Fell's [9, Lemma 4.2] implies that $\text{Ind}(T) \otimes V$ is unitarily equivalent to $\mathrm{Ind}(V|_H)$. In the following paragraph, we briefly present the details for the crossed product algebra case, and we shall be done.

Recall that the induced representation is modelled on the space $\tilde{\mathcal{H}}_L$ of Borel functions $f\colon G \to \mathcal{H}_L$ such that

$$(*) \qquad f(gh) = V(h)^{-1}f(g) \quad \text{and} \quad \int_{G/H} \|f(\bar{g})\|^2 \, d\bar{g} < \infty,$$

with $\langle \tilde{V}, \tilde{\pi} \rangle = \mathrm{Ind}(V|_H, \pi\rangle)$ defined by $(\tilde{V}(s)f)(g) = f(s^{-1}g)$ and $(\tilde{\pi}(a)f)(g) = \pi(g^{-1}a)f(g)$. Let c be a Borel cross-section for the quotient map of G onto G/H with $c(\bar{e}) = e$, so that each g in G can be written uniquely as $g = c(\bar{g})d(g)$, with $d\colon G \to H$ Borel. Let θ be a unitary operator from $\tilde{\mathcal{H}}_L$ into $L^2(G/H, \mathcal{H}_L)$ defined by $(\theta f)(\bar{g}) = V(c(\bar{g}))f(c(\bar{g}))$. Then

$$(\theta\tilde{V}(s)f)(\bar{g}) = V(c(\bar{g}))(\tilde{V}(s)f)(c(\bar{g})) = V(c(\bar{g}))f(s^{-1}c(\bar{g}))$$

while $((W \otimes V)(s)\theta f)(\bar{g}) = V(s)(\theta f)(\overline{s^{-1}g}) = V(s)V(c(\overline{s^{-1}g}))f(c(\overline{s^{-1}g}))$. As

$$(**) \qquad s^{-1}c(\bar{g}) = c(\overline{s^{-1}c(\bar{g})})d(s^{-1}c(\bar{g})) = c(\overline{s^{-1}g})d(s^{-1}c(\bar{g}))$$

by the cross-section property of c we have

$$\begin{aligned} V(c(\bar{g}))f(s^{-1}c(\bar{g})) &= V(c(\bar{g}))V(d(s^{-1}c(\bar{g}))^{-1})f(c(\overline{s^{-1}g})) && \text{by } (*) \\ &= V(s)V(c(\overline{s^{-1}g}))f(c(\overline{s^{-1}g})) && \text{by } (**). \end{aligned}$$

Likewise,

$$\begin{aligned} (\theta\tilde{\pi}(a)f)(\bar{g}) &= V(c(\bar{g}))\pi(c(\bar{g})^{-1}a)f(c(\bar{g})) \\ &= \pi(a)V(c(\bar{g}))f(c(\bar{g})) && \text{by the covariance of } \langle V, \pi \rangle \\ &= (I \otimes \pi)(a)(\theta f)(\bar{g}). \end{aligned}$$

It follows from the above Lemma that the hypothesis that G be discrete can be eliminated from Proposition 2.8 of [15]. We also have

COROLLARY 2.2. *Let (G, A, α) be a separable C^*-dynamical system, with G abelian, and let Γ denote the dual group of G. Each Γ-invariant ideal J of $G \times_\alpha A$ is of the form $G \times_\alpha I$, for a unique G-invariant ideal I of A.*

PROOF. $\mathrm{Res}\, J$ is a G-invariant ideal of A, by Proposition 11(i) of [12]. The Corollary now follows from Lemma 2.1, the fact that the maps Ex and Ind are identical, as mentioned earlier, and Proposition 12(i) of [12].

We denote by $\hat{A}$ the space of unitary equivalence classes of irreducible representations of A, and by $\mathrm{PR}(A)$ the space of primitive ideals of A. When both spaces are equipped with their usual hull-kernal topology, the action of G on A determines naturally jointly continuous actions of G on both $\hat{A}$ and $\mathrm{PR}(A)$. For Q in $\mathrm{PR}(A)$, the orbit closure $\overline{GQ} = \{P \in \mathrm{PR}(A): P \supseteq \bigcap_{g\in G} g \cdot Q\}$, and the quasi-orbit $\widetilde{GQ} = \{P \in \mathrm{PR}(A): \overline{GP} = \overline{GQ}\}$.

The space of quasi-orbits, endowed with the quotient topology relative to the map assigning to each P in $\mathrm{PR}(A)$ its quasi-orbit, is denoted by $(\mathrm{PR}(A)/G)^{\sim}$. It is well known [12, pages 221–222] that for a primitive ideal Q of $G \times_\alpha A$ and a representation $L = \langle V, \pi \rangle$ of $G \times_\alpha A$ with kernel $L = Q$, $\mathrm{Res}\, Q = \mathrm{kernel}\, \pi = \bigcap_{g \in G} g \cdot P$ for some primitive ideal P of A, and Q thus determines a unique quasi-orbit in $\mathrm{PR}(A)$. We say that Q "lives over" the quasi-orbit $\widetilde{GP}$ of P. The corresponding map $\theta: \mathrm{PR}(G \times_\alpha A) \to (PR(A)/G)^{\sim}$ assigning to Q in $\mathrm{PR}(G \times_\alpha A)$ the quasi-orbit over which Q lies is continuous [12, Proposition 9 and the Lemma on page 221], and, in case A is type I, onto by virtue of the "Mackey machine". We prove below two new results about the map θ: that it is onto even when A is not type I, by a simple application of Takai duality; and that it is open by virtue of Corollary 1.2. For both results we note that Res is Γ-invariant, since if $L = \langle V, \pi \rangle$ is a representation of $G \times_\alpha A$ and $\chi \in \Gamma$, then the representation χL corresponds to the covariant pair $\langle \chi V, \pi \rangle$, where $(\chi V)(g) = \chi(g) V(g)$, $g \in G$.

LEMMA 2.3. *Let (G, A, α) be a separable C^*-dynamical system, with G abelian. For each P in $\mathrm{PR}(A)$, there exists Q in $\mathrm{PR}(G \times_\alpha A)$ which "lives over" the quasi-orbit of P.*

PROOF. Let π be an irreducible representation of A on $\mathcal{H}_\pi$ with kernel $\pi = P$ and let $\langle V, \tilde{\pi} \rangle$ denote the representation of $G \times_\alpha A$ on $L^2(G, \mathcal{H}_\pi)$ induced from π. Defining a representation U of Γ, the dual of G, on $L^2(G, \mathcal{H}_\pi)$ by $(U(\chi)f)(t) = \chi(t) f(t)$, $\chi \in \Gamma$, $t \in G$, $f \in L^2(G, \mathcal{H}_\pi)$, it is almost immediate that U has the proper intertwining relationship with $\langle V, \tilde{\pi} \rangle$ so that the triple $\langle U, V, \tilde{\pi} \rangle$ defines a representation of $\Gamma \times_{\hat{\alpha}} (G \times_\alpha A)$, and also that this representation is irreducible. By the remarks preceding the Lemma, with A and G being replaced by, respectively, $G \times_\alpha A$ and Γ,

$$\mathrm{kernel}(\langle V, \tilde{\pi} \rangle) = \mathrm{Res}(\mathrm{kernel}(\langle U, V, \tilde{\pi} \rangle)) = \bigcap_{\gamma \in \Gamma} \gamma Q$$

for some primitive ideal Q of $G \times_\alpha A$. Upon applying the Res map once more, now as a map from ideals of $G \times_\alpha A$ to ideals of A, we have

$$\mathrm{Res}(\mathrm{kernel}(\langle V, \tilde{\pi}\rangle)) = \mathrm{kernel}\ \tilde{\pi} = \mathrm{Res}\left(\bigcap_{\gamma\in\Gamma} \gamma Q\right).$$

Now Res preserves intersections [12, Proposition 9] and is Γ-invariant, so $\mathrm{Res}(\bigcap_{\gamma\in\Gamma} \gamma Q) = \mathrm{Res}\ Q$. Thus $\mathrm{Res}\ Q$ equals kernel $\tilde{\pi}$, which is well-known to equal $\bigcap_{g\in G}\mathrm{kernel}(g\cdot\pi) = \bigcap_{g\in G} gP$, and we are done.

THEOREM 2.4. *Let (G, A, α) be a separable C^*-dynamical system, with G abelian. The map θ: $\mathrm{PR}(G\times_\alpha A)\to(\mathrm{PR}(A)/G)^\sim$, defined preceding Lemma 2.3, is continuous, open and onto.*

PROOF. By Lemma 2.3 and [12, Proposition 9 and the Lemma on page 221], we need only prove θ is open. If O is open in $\mathrm{PR}(G\times_\alpha A)$, so is $\Gamma O = \bigcup_{\gamma\in\Gamma} \gamma O$, and by the Γ-invariance of θ, noted immediately preceding Lemma 2.3, we may assume without loss of generality that O is an open, Γ-invariant subset of $\mathrm{PR}(G\times_\alpha A)$. By the definition of the hull-kernel topology, there exists a Γ-invariant ideal J of $G\times_\alpha A$ such that $O=\{P\in\mathrm{PR}(G\times_\alpha A)\colon P\not\supseteq J\}$. By Corollary 2.2 the ideal J is of the form $G\times_\alpha I$ for a G-invariant ideal I of A. As $G\times_\alpha I$ is generated by elementary tensors of the form $f\otimes i$, $f\in L^1(G)$, $i\in I$, and as, for a representation $L=\langle V,\pi\rangle$ of $G\times_\alpha A$, $L(f\otimes i)=\pi(i)V(f)$, it is clear that kernel $L\supseteq G\times_\alpha I$ if and only if kernel $\pi\supseteq I$. From this discussion and Lemma 2.3, $\theta(O)$ equals the set of quasi-orbits $\widetilde{GP}$ such that $\bigcap_{g\in G} gP\not\supseteq I$. By the G-invariance of I, this set is the image, in $(\mathrm{PR}(A)/G)^\sim$, of $\{P\in\mathrm{PR}(A)\colon P\not\supseteq I\}$. As the latter set is open in the hull-kernel topology and as the map of $\mathrm{PR}(A)$ onto $(\mathrm{PR}(A)/G)^\sim$ is open (the Lemma on page 221 of [12] again), we are done.

COROLLARY 2.5. *Let (G, A, α) be a separable C^*-dynamical system, and let Γ be the dual group of G. The spaces $(\mathrm{PR}(G\times_\alpha A)/\Gamma)^\sim$ and $(\mathrm{PR}(A)/G)^\sim$ are homeomorphic.*

PROOF. Consider the map $\theta\colon \mathrm{PR}(G\times_\alpha A)\to(\mathrm{PR}(A)/G)^\sim$ of Theorem 2.4. It is Γ-invariant, and as it is continuous and the space $(\mathrm{PR}(A)/G)^\sim$ is T_0 it is in fact constant on quasi-orbits, and thus induces a continuous map $\tilde{\theta}$ of $(\mathrm{PR}(G\times_\alpha A)/\Gamma)^\sim$ onto $(\mathrm{PR}(A)/G)^\sim$. The map $\tilde{\theta}$ is open since θ is, and we need only show that $\tilde{\theta}$ is one-to-one. Accordingly, let $L=\langle V,\pi\rangle$ and $R=\langle W,\tau\rangle$ be irreducible representations of $G\times_\alpha A$ with kernels, respectively, P and Q. If $\theta(P)=\theta(Q)$ then kernel π = kernel τ. Let U be the left regular representation of G on $L^2(G)$. As in the proof of Lemma 2.1, the representation $\langle U\otimes V, I\otimes\pi\rangle$ has kernel $\bigcap_{\chi\in\Gamma}\chi P$ and is unitarily equivalent to Ind π, while $\langle U\otimes W,$

$I \otimes \tau)$ has kernel $\bigcap_{\chi \in \Gamma} \chi Q$ and is unitarily equivalent to Ind τ. As induction is continuous, it follows that P and Q determine the same quasi-orbit in $\mathrm{PR}(G \times_\alpha A)$.

REMARKS. (1) A is G-simple if and only if $(\mathrm{PR}(A)/G)^{\sim}$ is a one point space. It thus follows from Corollary 2.5 that A is G-simple if and only if $G \times_\alpha A$ is Γ-simple. This well-known result [16, Lemma 6.1] was in fact the motivation for this section.

(2) It was shown in [19, Theorem 2.2], under certain special hypotheses implying among other things that A and $G \times_\alpha A$ are type I and that G acts trivially on $\hat{A}$, that not only is the map θ from $(G \times_\alpha A)^\wedge$ onto $\hat{A}$ continuous and open, but that $(G \times_\alpha A)^\wedge$ is indeed a principal Γ-bundle over $\hat{A}$.

(3) In the special case in which A is a commutative C^*-algebra, Theorem 2.4 follows from [21, Theorem 5.3].

§3. AF crossed product algebras

Our main goal in this section is to prove the following

THEOREM 3.1. *Let (G, A, α) be a separable C^*-dynamical system with G compact abelian and A type I AF. Then the crossed product C^*-algebra $G \times_\alpha A$ is also AF.*

The proof uses the following two lemmas. The first is somewhat stronger than we need, and what we need actually follows readily from Theorem 1 of [10]. However, Theorem 1 of [10] is deep and rather intricate, while in our case of a compact group action a short proof can be presented. We note that it is valid for any compact group G and any type I C^*-algebra A.

LEMMA 3.2. *Let (G, A, α) be a separable C^*-dynamical system, with G compact and A type I. Then A has a non-zero G-invariant liminal ideal I such that $\hat{I}$ is Hausdorff.*

LEMMA 3.3. *Let (G, A, α) be a separable C^*-dynamical system, with G compact abelian and A a liminal AF algebra with $\hat{A}$ Hausdorff. Then $G \times_\alpha A$ is AF.*

PROOF OF THEOREM 3.1. A standard transfinite argument will yield a composition series of G-invariant closed two-sided ideals $\{I_\rho\}_{0 \leq \rho \leq \tau}$ of A such that for each $\rho < \tau$, $I_{\rho+1}/I_\rho$ is a non-zero liminal G-invariant ideal of A/I_ρ such that $(I_{\rho+1}/I_\rho)^\wedge$ is Hausdorff. Clearly $G \times_\alpha A$ admits $\{G \times_\alpha I_\rho\}_{0 \leq \rho \leq \tau}$ as a composition series and it is enough to show that each $G\times_\alpha I_\rho$ is an AF algebra. This is done also by transfinite induction. The case of limit ordinal is obvious. Thus suppose that $\rho<\tau$ and $G\times_\alpha I_\rho$ is AF. $G\times_\alpha I_{\rho+1}$ is an extension of $G\times_\alpha I_\rho$ by $G\times_\alpha I_{\rho+1}/G\times_\alpha I_\rho$. This quotient to $*$-isomorphic to $G\times_\alpha I_{\rho+1}/I_\rho$ by [12, Proposition 12]. From Lemma 3.3 we infer that $G\times_\alpha I_{\rho+1}/I_\rho$ is an AF algebra. A well-known result on extensions of AF algebras ([5] and [8]) shows now that $G\times_\alpha I_{\rho+1}$ is an AF algebra.

Proof of Lemma 3.2. There is no loss of generality in supposing that A itself is liminal since the greatest liminal ideal of A is G-invariant. Now the conclusion is equivalent to the existence of a non-zero G-invariant open Hausdorff subset of $\hat{A}$.

Let O be a dense open Hausdorff subset of $\hat{A}$ and V a non-empty open subset of O such that $\bar{V}$, the closure of V in O, is compact. Clearly $G\bar{V}$ is compact too and from $G\bar{V}\subset GO$ we derive the existence of a finite set $\{g_i\}_{i=1}^n\subset G$ such that $G\bar{V}\subset\bigcup_{i=1}^n g_iO$. Put $W=V(\bigcap_{i=1}^n g_iO)$. By the density of O, W is a non-empty open subset of $\hat{A}$. Now GW is non-void, open and G-invariant. We want to show that GW is Hausdorff. Let π_1, π_2 be a pair of distinct points in GW which we want to separate by disjoint neighborhoods. Without loss of generality we may suppose that $\pi_1\in W$ and $\pi_2\in gW$ for some $g\in G$. Then $\pi_2\in gV$ so there is i, $1\leqq i\leqq n$, for which $\pi_2\in g_iO$. But then both π_1, π_2 belong to g_iO which is open and Hausdorff. Hence π_1, π_2 can be separated as needed.

Remark. After we had proven the above Lemma, we noticed Theorem 2.7 of [20], which proves that A in fact contains a non-zero G-invariant ideal of continuous trace. However, if V is an open subset of $\hat{A}$ corresponding to the dual of a continuous trace ideal, then it is easy ([6, Proposition 4.5.4]) to check that GV is the dual of a G-invariant continuous trace ideal provided one knows that GV is Hausdorff. Thus our result easily gives the stronger result. Also, apparently, a proof similar to ours appears in §8.1 of [18].

Before we tackle the proof of Lemma 3.3 we need one more lemma.

Lemma 3.4. *Let A be the C^*-algebra defined by a continuous field $((A(t)),\theta)$ of AF algebras over a totally disconnected Hausdorff space T. Then A is AF.*

Proof. The proof is actually given as part of the proof of the theorem on page 80 of [3], and we omit repeating the details.

Proof of Lemma 3.3. Let θ be the continuous open map of $(G\times_\alpha A)^\wedge$ onto $T=\hat{A}/G$, defined preceding Lemma 2.3. As $\hat{A}$ is Hausdorff and G is compact, clearly $\hat{A}/G$ is Hausdorff also. It follows from Theorem 4 of [14] that $G \times_\alpha A$ is $*$-isomorphic to the C^*-algebra defined by a continuous field of C^*-algebras over T. Moreover, the fiber algebra over $t \in T$ is $(G \times_\alpha A)/I_t$, where I_t is the intersection of all the primitive ideals in $\theta^{-1}(t)$. Thus, if $t = G\pi$ for some $\pi \in \hat{A}$, I_t is the intersection of all the kernels of the irreducible representations of $G \times_\alpha A$ which live over the orbit of π.

It is easy to see that T is totally disconnected since A is AF and $\hat{A}$ therefore has a basis of compact open sets. Thus, in view of the preceding lemma, it remains to show that each fiber algebra $(G \times_\alpha A)/I_t$ is AF. For t and π as above denote $J_t = \bigcap_{g \in G} \text{kernel}(g\pi)$. By the Γ-invariance of θ and Corollary 2.2 and its proof, it is clear that $J_t = \text{Res}(I_t)$ and that $I_t = G \times_\alpha J_t$. By Propositions 12 and 13 of [12] and the assumption that A is liminal, $(G \times_\alpha A)/I_t$ is $*$-isomorphic to $G \times_\alpha A/J_t$. By Theorem 2.13(ii) of [13], $G \times_\alpha A/J_t$ is $*$-isomorphic to $C^*(G'_\pi, \mathbf{C}, \tau') \otimes \mathcal{K}(\mathcal{H})$, where $C^*(G'_\pi, \mathbf{C}, \tau')$ is a certain twisted crossed product related to the C^*-dynamical system $(G_\pi, A/\text{kernel}\,\pi)$ as described on pages 218–219 of [12], and $\mathcal{K}(\mathcal{H})$ is the algebra of compact operators on the Hilbert space $\mathcal{H}$. Now G'_π is a compact topological group and $C^*(G'_\pi, \mathbf{C}, \tau')$ is a quotient of the group algebra $C^*(G'_\pi)$. As the latter is liminal with discrete spectrum [6, 15.1.5 and 18.4.3], it is a restricted direct sum of elementary algebras, hence AF, and we are done.

§4. Discrete abelian automorphism groups

In this section we discuss the action of discrete abelian groups.

THEOREM 4.1. *Let (G, A, α) be a separable C^*-dynamical system, with G discrete abelian. If both A and $G \times_\alpha A$ are type I, then $G \times_\alpha A$ is AF if and only if A is AF and, for every $\pi \in \hat{A}$, $\hat{G}_\pi$, the dual of the isotropy group G_π, is totally disconnected.*

We first prove

LEMMA 4.2. *Let (G, A, α) be a separable C^*-dynamical system with G discrete abelian and A liminal, with $\hat{A}$ homeomorphic to G/G_π for some $\pi \in \hat{A}$. Assume $G \times_\alpha A$ is type I. Then $G \times_\alpha A$ is AF if and only if $\hat{G}_\pi$ is totally disconnected.*

PROOF. Let $P = \text{kernel}\,\pi$. By Theorem 2.13 of [13] $G \times_\alpha A$ is isomorphic to $C^*(G'_\pi, \mathbf{C}, \tau') \otimes \mathcal{K}(\mathcal{H})$, $\mathcal{H}$ a certain Hilbert space and $C^*(G'_\pi, \mathbf{C}, \tau')$ the Mackey system associated to $(G_\pi, A/P, \alpha)$. As $C^*(G'_\pi, \mathbf{C}, \tau')$ is isomorphic to a hereditary subalgebra of $C^*(G'_\pi, \mathbf{C}, \tau') \otimes \mathcal{K}(\mathcal{H})$, $G \times_\alpha A$ is AF if and only if $C^*(G'_\pi, \mathbf{C}, \tau')$ is, by Theorem 3.1 of [8]. Assuming, as we do, that $G \times_\alpha A$ is type I, so is $C^*(G'_\pi, \mathbf{C}, \tau')$. Note that G'_π can be regarded as the central extension of T by G_π determined by a cocycle a on G_π. Letting

$$S = \{s \in G_\pi : a(s, t) = a(t, s) \text{ for all } t \in G_\pi\},$$

it is known that a is cohomologous to a cocycle lifted from a totally skew cocycle b of G_π / S (Theorem 3.1 of [1]). Let Z denote the center of G'_ρ, and $Z^\perp$ the annihilator of $Z/N_{\tau'}$ in $G'_\pi/N_{\tau'}$. It follows from the discussion on pages 218–219 of [12] that $G'_\pi/N_{\tau'}$ can be naturally identified with G_π, and one can easily check that $Z/N_{\tau'}$ identifies with S. Hence as $(G'_\pi/N_{\tau'})^\wedge/Z^\perp$ identifies with the dual of $Z/N_{\tau'}$, or with $\hat{S}$, it follows from Proposition 34 of [12] that $C^*(G'_\pi, \mathbf{C}, \tau')^\wedge$ is homeomorphic to $\hat{S}$. As $C^*(G'_\pi, \mathbf{C}, \tau')$ is type I, it is AF if and only if its dual $C^*(G'_\pi, \mathbf{C}, \tau')^\wedge$ has a basis of compact open sets [3, page 80]. We shall be done once we check that $\hat{S}$ is totally disconnected if and only if $\hat{G}_\pi$ is. From our hypothesis that $G \times_\alpha A$ be type I, again, it follows that the totally skew cocycle b of G_π / S is type I, and thus, by Lemma 3.1 of [1], that G_π / S is finite. Thus $S^\perp$, the annihilator of S in $\hat{G}_\pi$, is finite, and as $\hat{G}_\pi / S^\perp$ is naturally homeomorphic to $\hat{S}$, the result is clear.

Remark. It follows (page 80 of [3]) from our hypotheses that A be liminal with $\hat{A}$ homeomorphic to the discrete space G/G_π, that A is automatically AF in Lemma 4.2.

Proof of Theorem 4.1. First assume that A and $G \times_\alpha A$ are type I, and that G is discrete abelian. By Theorem 3.2 of [11], G acts smoothly on $\hat{A}$. As the largest liminal ideal J of A is G-invariant and G acts smoothly on the open subset $\hat{J}$ of $\hat{A}$, we may apply Theorem 1 of [10] and a standard composition series argument, as in the proof of Theorem 3.1, to find a composition series $\{I_\rho\}_{0 \leq \rho \leq \tau}$ of G-invariant closed two-sided ideals of A such that for every $\rho < \tau$, $I_{\rho+1}/I_\rho$ is a non-trivial liminal G-invariant ideal of A/I_ρ and the orbit space $(I_{\rho+1}/I_\rho)^\wedge/G$ is Hausdorff. Note that we are not claiming here that $(I_{\rho+1}/I_\rho)^\wedge$ itself is Hausdorff. Note also that by Theorem 3.2 of [11] again, G does act smoothly on $(I_{\rho+1}/I_\rho)^\wedge$.

Assume now in addition that A is AF and that for each $\pi \in \hat{A}$, $\hat{G}_\pi$ is totally disconnected. To show $G \times_\alpha A$ is AF it suffices, exactly as in the proof of Theorem 3.1, to check that each $G \times_\alpha (I_{\rho+1}/I_\rho)$ is AF. Changing notation, we may assume that A is AF liminal with $\hat{A}/G$ Hausdorff, and that $G \times_\alpha A$ is type I. To check that $G \times_\alpha A$ is AF, apply Theorem 4 of [14] and Theorem 2.4 to conclude that $G \times_\alpha A$ is $*$-isomorphic to the C^*-algebra defined by a continuous field of C^*-algebras over $T = \hat{A}/G$. As $\hat{A}$ has a basis of compact open sets and as the map of $\hat{A}$ onto $\hat{A}/G$ is continuous and open, $\hat{A}/G$ clearly also has a basis of compact open sets. Exactly as in the proof of Lemma 3.3, the fiber algebras are of the form $G \times_\alpha (A/J)$, to which Lemma 4.2 applies. Thus, by Lemma 3.4, $G \times_\alpha A$ is indeed AF.

To prove the converse, suppose now that $G \times_\alpha A$ is AF, with G discrete abelian and A and $G \times_\alpha A$ type I, as before. The dual group Γ of G is compact abelian, and by Theorem 3.1, $\Gamma \times_{\hat{\alpha}} (G \times_\alpha A)$ is AF. This algebra is $*$-isomorphic, by Takai duality, to $A \otimes \mathcal{K}(L^2(G))$, and thus A, which is $*$-isomorphic to a hereditary subalgebra of $A \otimes \mathcal{K}(L^2(G))$, is AF also by Theorem 3.1 of [8]. Let $\pi \in \hat{A}$. By Theorem 1 of [10] the orbit $G\pi$ in $\hat{A}$ is open in its closure, and homeomorphic to G/G_π. Thus there are G-invariant ideals I, J of A with $J \supseteq I$ such that $(J/I)^\wedge$ is homeomorphic to $G\pi$. As $G \times_\alpha (J/I)$ is isomorphic to $(G \times_\alpha J)/(G \times_\alpha I)$, it is AF and type I. Furthermore J/I is postliminal with discrete spectrum, hence liminal [6, Problem 4.7.15]. Thus Lemma 4.2 applies and we are done.

Remark. It has been shown in [4, Proposition 4.3.2] that the crossed product of a unital C^*-algebra by $\mathbf{Z}$ is never AF. Thus it follows from this result and Theorem 4.1 that a unital type I AF algebra admits no free and smooth action by $\mathbf{Z}$.

Added in proof. Since submitting the paper for publication, the authors were able to remove the commutativity hypothesis for the group G in Theorems 2.4 and 3.1. The details will appear elsewhere.

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